AN EMPIRICAL EVALUATION OF NUMERICAL METHODS FOR AMERICAN PUT OPTION VALUATION

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INTRODUCTION

An option is a financial instrument that provides various benefits to investors and traders. An American put option is a contract that is sold by one party (the option writer) to another party (the option holder). The contract has a prescribed maturity date, a prescribed underlying asset, and a prescribed strike price. The option holder has the right, but not the obligation, to sell the underlying asset to the option writer at the strike price any time up to and including the maturity date of the option. To solve for the fair market price for an American put option, we must decide when it is optimal to exercise the option. This pricing problem can be formulated mathematically as a free-boundary-value problem for a partial differential equation (PDE).

OBJECTIVES

$\mathcal{L}V(S,t) = 0$ is the famous *Black-Scholes* (or *Black-Scholes-Merton*) PDE, where

$$\mathcal{L}V(S,t) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

The final condition and boundary conditions are

$$V(S,T) = P(S) = \max(E - S, 0),$$
$$V(0,t) = E, V(S,t) \to 0 \text{ as } S \to \infty$$
$$V \text{ and } \frac{\partial V}{\partial S} \text{ are continuous}$$

The objectivity is to solve following *discretized linear complementary problem*:

$$\mathcal{L}V \geq 0 \,, V - P \geq 0 \,, \mathcal{L}V \cdot (V - P) = 0$$

FINITE-DIFFERENCE METHOD

Let u_n^m be an approximation to the "transformed" price of the option at time t_m and $x_n = log(S_n/E)$, where S_n is the associated stock price and E is the strike price. To solve for u_n^m, we need to apply a numerical method to the transformed Black-Scholes PDE. The numerical methods that we consider in this poster are the following finite-difference methods.

- Explicit Finite-difference Method

$$u_n^{m+1} = \lambda u_{n-1}^m + (1 - 2\lambda)u_n^m + \lambda u_{n+1}^m$$

- Fully Implicit Finite-difference Method (FIFDM)

$$-\lambda u_{n-1}^{m+1} + (1 + 2\lambda)u_n^{m+1} - \lambda u_{n+1}^{m+1} = u_n^m$$

- Crank-Nicolson Method (CNFDM)

$$Z_n^m = (1 - \lambda)u_n^m + \frac{\lambda}{2}(u_{n-1}^m + u_{n+1}^m)$$
$$(1 + \lambda)u_n^{m+1} - \frac{\lambda}{2}(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = Z_n^m$$

- Rannacher Smoothing Method
  1. Run FIFDM for the first few timesteps.
  2. Then switch back to CNFDM.

where $\lambda = \frac{\delta\tau}{\delta x^2}$.

NUMERICAL TECHNIQUES

1. Projected Succssive Over-Relaxation:
Start with an initial guess and repeat the following iteration until it converges:

$$y_n^{m+1,k+1} = \frac{1}{1 + 2\lambda}(b_n^m + \lambda(u_{n+1}^{m+1,k} + u_{n-1}^{m+1,k+1})$$
$$\epsilon = y_n^{m+1,k+1} - u_n^{m+1,k}$$
$$u_n^{m+1,k+1} = \max(u_n^{m+1,k} + \omega\epsilon, \mathbf{g}_n^{m+1})$$
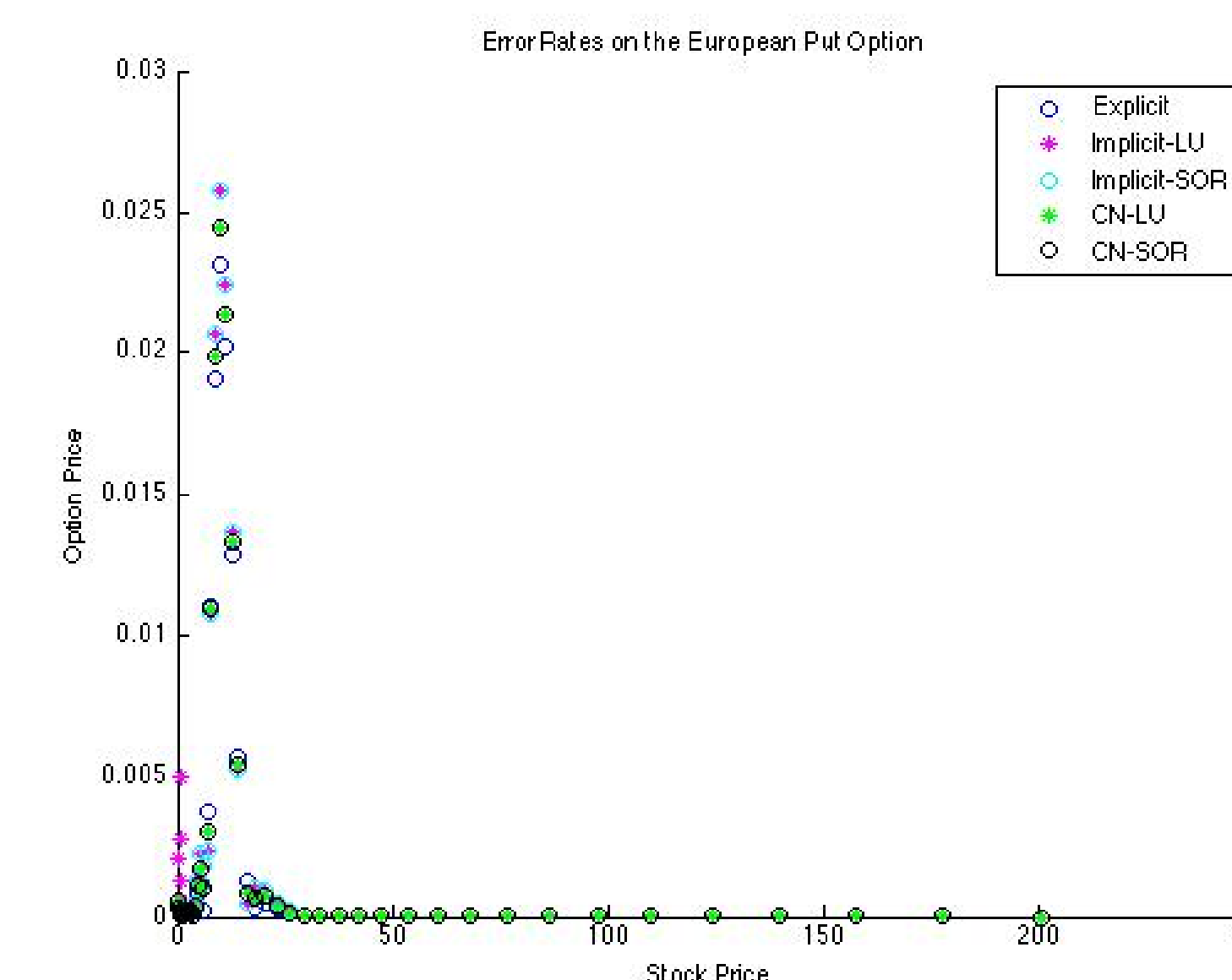
2. The penalty method:
The penalty method approximately satisfies the constraints by adding a forcing term to the right side of the PDE at the points for which the constraints are not satisfied. This makes the PDE non-linear. We use a Newton iteration to solve for the solution values at each timestep.

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} = (1 - \theta)\mathcal{L}u_n^{m+1} + \theta\mathcal{L}u_n^m + p_n^{m+1}$$

$$p_n^{m+1,k} = \begin{cases} \frac{g_n^m - u_n^{m+1,k}}{\delta\tau \cdot L}, & \text{if } \mathbf{u}_n^{m+1,k} < \mathbf{g}_n^m \\ 0, & \text{if } \mathbf{u}_n^{m+1,k} \geq \mathbf{g}_n^m \end{cases}$$

3. Improved PSOR:

  1. Perform Projected SOR
  2. Reduce-Space
  3. Repeat until free boundary converges

EXPERIMENTS

$\delta\tau$	$\delta\mathbf{x_1}$	$\delta\mathbf{x_2}$	$\delta\mathbf{x_3}$	I	CN	RS	ICN	II	IRS
0.0004	0.15	0.075	0.0375	4.0944	4.1165	4.0980	4.1158	4.2050	4.1132
0.0004	0.12	0.060	0.0300	4.1845	4.1970	4.1915	4.2659	3.7509	4.2228
0.0004	0.075	0.0375	0.01875	3.9247	3.9861	3.9708	3.1821	2.7145	2.9424
0.0001	0.075	0.0375	0.01875	3.9035	3.9301	3.9151	4.2043	3.8704	4.1984

Table 1: Error Validation of an American put option evaluated with respect to $O(\delta x^2)$ by different methods

$\delta\tau$	$\delta\mathbf{S_1}$	$\delta\mathbf{S_2}$	$\delta\mathbf{S_3}$	**Implicit-Penalty**	**CN-Penalty**
0.0017	2.5	1.25	0.625	4.5213	4.5429
0.0017	2	1	0.5	3.9739	3.9416
0.0017	1	0.5	0.25	4.1351	3.9247
0.0001	0.075	0.0375	0.01875	3.7679	3.9035

Table 2: Error Validation of an American Put Option evaluated with respect to $O(\delta x^2)$ by the penalty method

M	N	II	ICN	IRS	**Imp-Penalty**	**CN-Penalty**
10	500	0.080366	0.071208	0.070966	0.021927	0.023686
10	600	0.138788	0.095581	0.095268	0.089753	0.067151
10	1000	0.333361	0.320676	0.318319	0.041413	0.072715

Table 3: CPU time required by each method to compute the American Put Option Price

Figure 1: Error rate for an European Option. Figure 1 illustrates the where the most errors comes from.
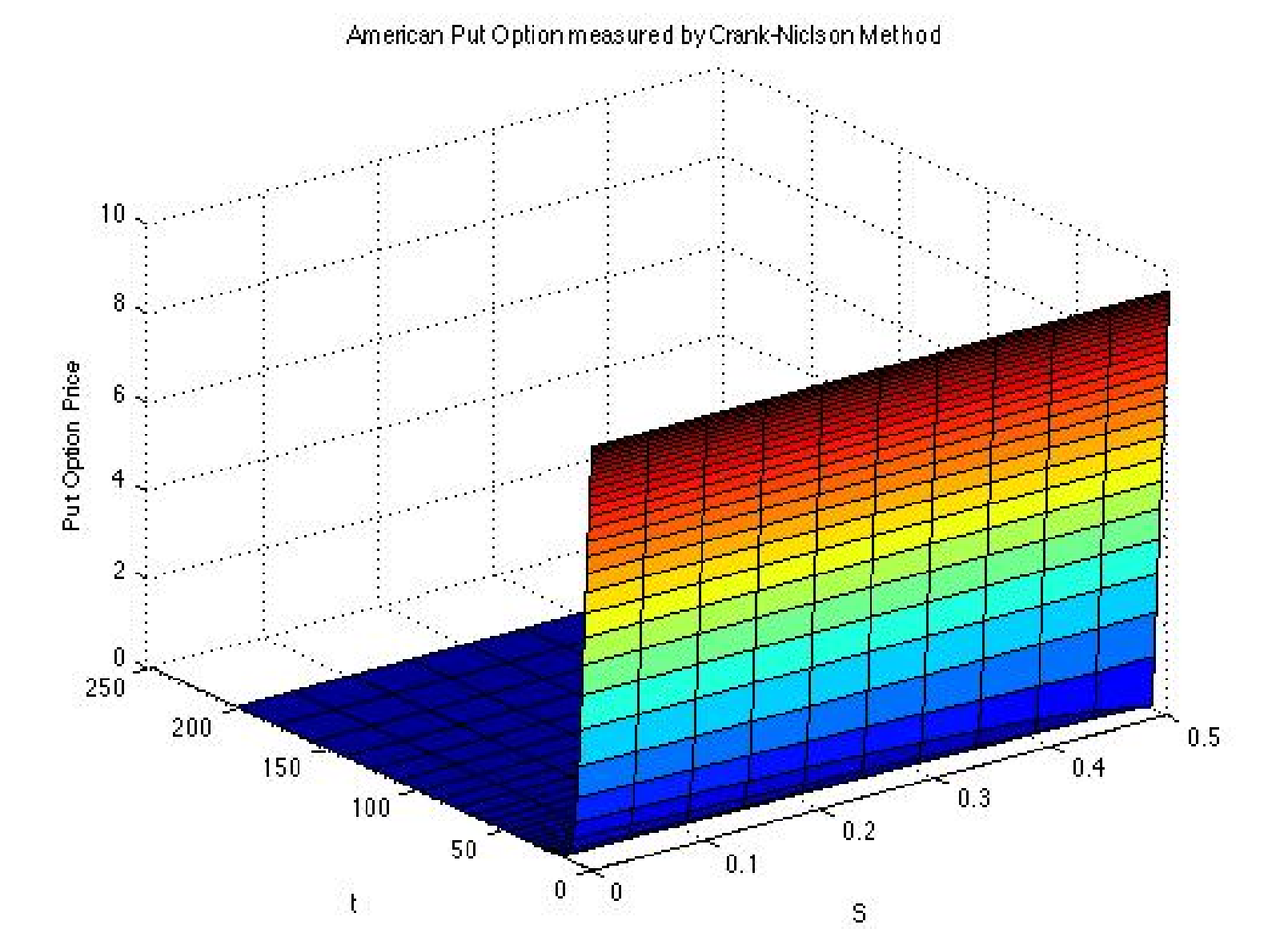
Figure 2: American Put Option Price

ERROR MEASURE

To verify the correctness of the implementation, we evaluate the following formula to obtain an approximation to the rate of convergence of the method.

We ought to get either approximately 4 or 2 depending on whether the rate of convergence is quadratic or linear, respectively.

$$\frac{err_1 - err_2}{err_2 - err_3}$$